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Approximation of Extremal Functions in H^p by an Iterative Method[1]

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For $L^* = \{f: f \in H^p; f(z_k) = w_k, k = 1, 2,..., m, | z_k | < 1, \text{and } | w_k | < 1\}$, Macintyre, Rogosinski, and Shapiro showed that the function f^* of L^* which minimizes

$$\|f\|_p = \left\{\int_{|z|=1} |f(z)|^p \, | dz |\right\}^{1/p}$$

is of the form

$$C \prod_{i=1}^{K} [(z - c_i)/(1 - \bar{c}_i z)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

where $| c_i | < 1$ and $K \leqslant m - 1$. The present paper suggests approximation of f^* by an iterative method using polynomials $P_n(z)$ of degree $m - 1$. A corresponding sequence of functions (f_n) is defined in such a way that, under appropriate hypotheses, (f_n) converges to f^*. To define f_n from f_{n-1}, an adjustment is made in the values w_{kn} to which the polynomial P_n is required to interpolate at the fixed points z_k. The method appears to be appropriate for approximation of a function which is of the same form as f^* and assumes the given values w_k at the designated points z_k. By Theorem 1, such a function is f^*. A Computer Science student at Purdue is attempting to program the procedure.[1]

For given p, $1 \leqslant p \leqslant \infty$, $H^p = \{f: f$ is analytic for $| z | < 1$, and $\|f(rz)\|_p$ is bounded for $0 < r < 1\}$. On $| z | = 1$, f is defined almost everywhere as $\lim_{r \to 1-} f(re^{i\theta})$ where $z = re^{i\theta}$. Then $H^p \subset L^p$, with the norm

[1] A sequel to this paper "Determination of Extremal Functions in H^p by a Fortran Program," to appear in *SIAM J. Numer. Anal.*, reports on success of the Fortran Program for this method in all cases tried, yielding 60 extremal functions.

$\|f\|_p = \{\int_{|z|=1} |f|^p \,|\, dz \,|\}^{1/p} < \infty$. The closed convex subset L^* of L^p has an essentially unique element f^* such that $\|f^*\|_p \leqslant \|f\|_p$, $f \in H^p$. In this paper f^* is referred to as the *minimizing function* or *element* of L^*. In the definition of L^*, some of the z_k are allowed to coincide, with the understanding that appropriate w_k are to be regarded as values assigned to derivatives at those z_k.

1. Equivalent Problems

We let Q be the polynomial of degree $m - 1$ such that $Q(z_j) = w_j$, $j = 1,..., m$, and let B denote the Blaschke product

$$B(z) = \prod_{j=1}^{M} (z - z_j)/(1 - \bar{z}_j z),$$

where $M \leqslant m - 1$.

If $h \in H^p$, then $[Q(z) + h(z) \prod_{j=1}^{m} (z - z_j)/(1 - \bar{z}_j z)] \in L^*$. On the other hand, if $f \in L^*$, then $(f - Q)/B \in H^p$. For reference we state

Lemma 1.1. *f^* minimizes $\|f\|_p$, $f \in L^*$, if and only if $\|(-Q/B) - h\|_p$, $h \in H^p$, is minimized by $h^* = (+f^* - Q)/B$.*

The solution of any one of the following four problems yields a solution to the others. (See the references of Rogosinski, Macintyre, and Shapiro.) We set $(1/p) + (1/q) = 1$.

I. Determine $f^* \in L^*$ such that

$$\|f^*\|_p \leqslant \|f\|_p, \qquad f \in L^*.$$

II. Determine $h^* \in H^p$ such that

$$\|(-Q/B) - h^*\|_p \leqslant \|(-Q/B) - h\|_p, \qquad h \in H^p.$$

III. Determine $g^* \in H^q$ such that

$$\left| (1/2\pi i) \int [(-Q(z)/B(z)] \, g(z) \, dz \right|$$

is maximal, for $g \in H^q$, and $\| g \|^q = 1$.

IV. Determine C and c_i, $i = 1,..., m - 1$, $| c_i | \leqslant 1$, such that

$$C[\Pi'(z_k - c_i)/(1 - \bar{c}_i z_k)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z_k)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z_k)^{2/p} = w_k,$$
$$k = 1,..., m,$$

where the product Π' is over some subset of $\{1, 2,..., m - 1\}$.

According to the above mentioned references, answers to the above problems are known to be of the following forms thus related:

$$[(-Q(z)/B(z)) - h^*(z)] \prod_{j=1}^{m} (z - z_j)/(1 - \bar{z}_j z)$$

$$= -f^*(z) = C\Pi'[(z - c_i)/(1 - \bar{c}_i z)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p}.$$

For Problem III,

$$g^*(z) = A\Pi''[(z - c_i)/(1 - \bar{c}_i z)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z)^{2/q} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/q},$$

where Π' and Π'' are complementary products of the Blaschke product $\prod_{i=1}^{m-1} (z - c_i)/(1 - \bar{c}_i z)$, that is, each of the factors $(z - c_i)/(1 - \bar{c}_i z)$ is in exactly one of the products.

In this paper we describe a method of obtaining a sequence of functions (f_n), containing a subsequence (f_{n_ν}) which converges to $f^*(z)$, the solution of Problem I, provided $\lim_{\nu\to\infty} f_{n_\nu}(z_k) = w_k$, $k = 1, 2, \ldots, m$.

Lemma 1.2. *Let*

$$\tilde{f} = D \prod_{i=1}^{N} [(z - d_i)/(1 - \bar{d}_i z)] \prod_{i=1}^{m-1} (1 - \bar{d}_i z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

with $|d_i| < 1$, *and let* $w_j = \tilde{f}(z_j)$, $j = 1, \ldots, m$, *where* D *is chosen so that* $|w_j| < 1$. *Let* f^* *be the minimizing function for* L^* *determined by the* z_j *and the corresponding* w_j. *Then* $\tilde{f} \equiv f^*$.

Proof. Define

$$\tilde{g}(z) = \left(1 \Big/ \left\| \left[\prod_{i=1}^{m-1} (1 - \bar{d}_i z) \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z) \right]^{2/q} \right\|_q \prod_{i=N+1}^{m-1} (z - d_i)/(1 - \bar{d}_i z) \right)$$

$$\times \left(\prod_{i=1}^{m-1} (1 - \bar{d}_i z)^{2/q} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/q} \right),$$

and let $\lambda(z) = z\tilde{g}(z)[\tilde{f}(z) \prod_{j=1}^{m} (1 - \bar{z}_j z)/(z - z_j)]$. Then

$$\lambda(z) = C \left[z \prod_{i=1}^{m-1} (z - d_i) \prod_{i=1}^{m-1} (1 - \bar{d}_i z) \right] \Big/ \left[\prod_{j=1}^{m} (z - z_j) \prod_{j=1}^{m} (1 - \bar{z}_j z) \right].$$

Noting that $\lambda(1/\bar{z}) = \overline{\lambda(z)}$, we obtain that (1) $\lambda(z)$ is real on $|z| = 1$, and (2) $|\tilde{g}(z)|^{1/p}$ and $|\tilde{f}(z)\,\Pi[(1 - \bar{z}_j z)/(z - z_j)]|^{1/q}$ are proportional almost everywhere on $|z| = 1$. Now (1) and (2) imply that equality holds in Hölder's Inequality [7, pp. 282–3]; hence

$$\left| \int_{z=e^{i\theta}} \tilde{g}(z) \tilde{f}(z) \prod_{j=1}^{m} [(1 - \bar{z}_j z)/(z - z_j)] \, dz \right| = \| \tilde{g} \|_q \| \tilde{f} \|_p = \| \tilde{f} \|_p .$$

We have

$$\| f^* \|_p = \max_{\|g\|_q = 1} \left| \int g[\tilde{f}/B] \, dz \right| \geqslant \left| \int \tilde{g}[\tilde{f}/B] \, dz \right| = \| \tilde{f} \|_p .$$

(For the first equality, see [12, Theorem 1].) Since f^* is minimal for L^*, this yields $\| \tilde{f} \|_p = \| f^* \|_p$. By the uniqueness of the minimizing element for $L^*, \tilde{f} = f^*$.

As an immediate consequence of the lemma, we have

THEOREM 1. *If a function f of the form*

$$C \prod_{i=1}^{N} [(z - c_i)/(1 - \bar{c}_i z)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

for some $N \leqslant m - 1$, *assumes the assigned values* w_j *at the corresponding* z_j, *then f is the minimizing function* f^* *for* L^*.

2. Development of an Iterative Method for problem IV

The polynomial P_1 of degree $m - 1$ satisfying $P_1(z_k) = u_{k1} = w_k d_k^{2/p}$, where $d_k = \prod_{j=1}^{m} (1 - \bar{z}_j z_k)$, is uniquely determined; in fact, by the Lagrange interpolation formula, $P_1(z_k) = \omega(z) \sum_{k=1}^{m} (u_{k1}/\omega'(z_k))/(z - z_k)$, where $\omega(z) = \prod_{j=1}^{m} (z - z_j)$. Now $P_1(z)$ may be written in the form

$$C_1 \prod_{i=1}^{K_1} ((z - c_{i1})/(1 - \bar{c}_{i1} z)) \prod_{i=1}^{m-1} (1 - \bar{c}_{i1} z),$$

where $c_{11}, c_{21}, \ldots, c_{K_1 1}$ are those zeros of P_1 lying in $|z| \leqslant 1$. We let $B_{K_1 1}^{(c)} = B_1^{(c)}$ denote $\prod_{i=1}^{K_1} (z - c_{i1})/(1 - \bar{c}_{i1} z)$. For

$$f_1(z) = C_1 B_{K_1}^{(c)}(z) \prod_{i=1}^{m-1} (1 - \bar{c}_{i1} z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

we have $f_1(z_k) = w_k \prod_{i=1}^{m-1} (1 - \bar{c}_{i1} z_k)^{(2/p)-1}$.

Subsequent steps are outlined in the following paragraphs. After new values u_{k2} are substituted for the $u_{k1}(= w_k d_k^{2/p}$, above), the procedure of the preceding paragraph is repeated, using a polynomial P_2, of the same degree $m-1$ as P_1, to define a corresponding $f_2(z)$.

For P_2, the polynomial of degree $m-1$ satisfying $P_2(z_k) = u_{k2} = w_k d_k^{2/p} / \prod_{i=1}^{m-1} (1 - \bar{c}_{i1} z_k)^{(2/p)-1}$, we write

$$P_2(z) = C_2 B_2^{(c)}(z) \prod_{i=1}^{m-1} (1 - \bar{c}_{i2} z)$$

and define

$$f_2(z) = C_2 B_2^{(c)}(z) \prod_{i=1}^{m-1} (1 - \bar{c}_{i2} z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p}.$$

Then

$$f_2(z_k) = w_k \left[\prod_{i=1}^{m-1} (1 - \bar{c}_{i2} z_k) \Big/ \prod_{i=1}^{m-1} (1 - \bar{c}_{i1} z_k) \right]^{(2/p)-1}.$$

Inductively, we define

$$f_n(z) = C_n B_n^{(c)}(z) \prod_{i=1}^{m-1} (1 - \bar{c}_{in} z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

where

$$B_n^{(c)}(z) = \prod_{i=1}^{K_n} [(z - c_{in})/(1 - \bar{c}_{in} z)]$$

is determined by the polynomial

$$P_n(z) = C_n \prod_{i=1}^{K_n} [(z - c_{in})/(1 - \bar{c}_{in} z)] \prod_{i=1}^{m-1} (1 - \bar{c}_{in} z)$$

satisfying

$$P_n(z_k) = w_k d_k^{2/p} \Big/ \left[\prod_{i=1}^{m-1} (1 - \bar{c}_{i,n-1} z_k) \right]^{(2/p)-1}.$$

Then

$$f_n(z_k) = w_k \left[\prod_{i=1}^{m-1} (1 - \bar{c}_{in} z_k) \Big/ \prod_{i=1}^{m-1} (1 - \bar{c}_{i,n-1} z_k) \right]^{(2/p)-1}.$$

3. Sufficient Conditions for the Iterative Method to Yield the Solution of Problem IV

If $(c_{in})_{n=1}^{\infty}$, $i = 1,..., m-1$, and $(C_n)_{n=1}^{\infty}$ converge, we denote the respective limits by c_i and C; however, if a convergent subsequence is being considered, c_i and C denote the respective limits of the subsequences in question, $(c_{in_\nu})_{\nu=1}^{\infty}$ and $(C_{n_\nu})_{\nu=1}^{\infty}$. We define ρ as min $|(1/\bar{c}_i)|$, where the minimum is over those i such that $|c_i| < 1$.

We note that each of the sequences

$$\left(\left[\prod_{i=1}^{m-1} (1 - \bar{c}_{in} z_k) \Big/ \prod_{i=1}^{m-1} (1 - \bar{c}_{i,n-1} z_k)\right]^{(2/p)-1}\right)_n, \qquad k = 1,..., m,$$

is bounded by $[(1 + R)/(1 - R)]^{|(m-1)((2/p)-1)|}$, where $R = \max_j |z_j|$, and each of the sequences $(c_{in})_{n=1}^{\infty}$, $i = 1,..., m - 1$, is bounded by 1. Hence, there exists an $(n_\nu)_{\nu=1}^{\infty}$ such that the corresponding subsequence of each of the above sequences converges. Also each subsequence $(B_{n_\nu}^{(c)}(z))_\nu$ of $(B_n^{(c)}(z))_n$ is uniformly bounded on compact subsets of the disk $|z| < \rho$. Applying the Ascoli–Arzela Theorem, we obtain

LEMMA 2.1. *There exists a subsequence (n_ν) of (n) such that, as ν tends to infinity, each of the corresponding subsequences of the sequences designated in the preceding paragraph converges, also such that*

$$B_{n_\nu}^{(c)}(z) = \prod_{i=1}^{K_n} (z - c_{i,n_\nu})/(1 - \bar{c}_{in_\nu} z)$$

converges as $\nu \to \infty$, uniformly on compact subsets of the disk $|z| < \rho$.

LEMMA 2.2. *If some subsequence of $(f_n(z))$ converges at some point z_0 interior to $|z| = \rho$, then some subsequence (f_{n_ν}) of (f_n) converges to a function of the form*

$$\tilde{f}(z) = C \prod_{i=1}^{K} [(z - c_i)/(1 - \bar{c}_i z)] \prod_{i=1}^{m-1} (1 - \bar{c}_i z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p}.$$

The convergence is uniform on compact subsets of the disk $|z| < \rho$, and

$$\tilde{f}(z_k) = \lim_{\nu \to \infty} \left[\prod_i (1 - \bar{c}_{i,n_\nu} z_k)^{(2/p)-1} \Big/ \prod_i (1 - \bar{c}_{i,n_\nu - 1} z_k)^{(2/p)-1}\right] w_k .$$

Proof. There exists a subsequence (n_ν) of (n) such that each of the subsequences of Lemma 2.1 converges. Thus, a convergent subsequence of $(f_n(z_0))$ is determined, and the corresponding $(C_{n_\nu})_{\nu=1}^{\infty}$ converges.

LEMMA 2.3. *If $\prod_{i=1}^{m-1} (1 - \bar{c}_{in_\nu} z_k)/\prod_{i=1}^{m-1} (1 - c_{i,n_\nu - 1} z_k)$ converges to 1 for $k = 1, 2,..., m$, then some subsequence of $(f_n(z))$ converges to $f^*(z)$, uniformly on compact subsets of the disk $|z| < \rho$.*

Proof. The hypothesis, combined with Lemma 2.2, yields that some subsequence converges to

$$C\left[\prod_{i=1}^{K}(z-c_i)/(1-\bar{c}_i z)\right]\left[\prod_{i=1}^{m-1}(1-\bar{c}_i z)\right]^{2/p}\Big/\left[\prod_{j=1}^{m}(1-\bar{z}_j z)\right]^{2/p},$$

which takes on the assigned values w_k at the designated z_k. By Theorem 1, this is just $f^*(z)$.

THEOREM 2. *If* $\lim_{n\to\infty} f_n(z_k) = w_k$, $k = 1,..., m$, *then* $\lim_{n\to\infty} f_n(z) = f^*(z)$.

Proof. Since

$$\lim_{n\to\infty} f_n(z_k) = w_k \lim_{n\to\infty}\left[\prod_{i=1}^{m-1}(1-\bar{c}_{in}z_k)\Big/\prod_{i=1}^{m-1}(1-\bar{c}_{i,n-1}z_k)\right]^{2/p},$$

evidently, $\lim_{n\to\infty}[\prod_{i=1}^{m-1}(1-\bar{c}_{in}z_k)/\prod_{i=1}^{m-1}(1-\bar{c}_{i,n-1}z_k)] = 1$. Lemma 2.3 yields the conclusion.

THEOREM 3. *If* $\lim_{n\to\infty} c_{in}$ *exists for* $i = 1,..., m-1$, *then* $\lim_{n\to\infty} f_n(z) = f^*(z)$.

Proof. Since

$$\lim f_n(z_k) = w_k \lim_{n\to\infty}\left[\prod_{i=1}^{m-1}(1-\bar{c}_{in}z_k)\Big/\prod_{i=1}^{m-1}(1-\bar{c}_{i,n-1}z_k)\right] = w_k,$$

the hypothesis of the preceding theorem is satisfied.

4. DISCUSSION OF METHOD

In the case $p = 2$, the function f_1 obtained in the first step of the procedure outlined above is actually f^* [14, pp. 147, 227]. Moreover, $P_1(z)$, the polynomial of degree $m - 1$ used there, actually minimizes

$$\int_{|z|=1}\left|f^*(z) - P(z)\Big/\left[\prod_{j=1}^{m}(1-\bar{z}_j z)\right]\right|^2 |dz|, \qquad P \in H_2. \qquad (*)$$

For, with $|1/\prod_{j=1}^{m}(1-\bar{z}_j z)|$ as the weight function [14, Theorem 2, p. 147], the essentially unique function P of H_2 which minimizes (*) is the polynomial

$P_1(z)$ of degree $m-1$ which interpolates to $f^*(z)\prod_{j=1}^{m}(1-\bar{z}_j z)$ at the z_k. We note that P_n minimizes

$$\int_{|z|=1}\left|\left[f^*(z)\prod_{i=1}^{m-1}(1-\bar{c}_{in}z)\Big/\prod_{i=1}^{m-1}(1-\bar{c}_{i,n-1}z)\right]-P(z)\Big/\prod(1-\bar{z}_j z)\right|^2 |dz|,$$

$$P \in H_2.$$

It is of some interest to reverse our procedure, that is, to begin with the function f^* and, by an analogous procedure to attempt to obtain the polynomial Q of degree $m-1$ satisfying $Q(z_j)=w_j$, $j=1,..., m$.

Let z_k, w_k, and L^* be as in Section 1. Let

$$\begin{aligned} f^*(z) &= g_1(z) \\ &= A_1\prod_{i=1}^{M_1}[(z-a_{i1})/(1-\bar{a}_{i1}z)]\left[\prod_{i=1}^{m-1}(1-\bar{a}_{i1}z)\right]^{2/p}\Big/\left[\prod_{j=1}^{m}(1-\bar{z}_j z)\right]^{2/p}, \end{aligned}$$

where $|a_{i1}|<1$, $i=1,..., m-1$, and $g_1(z_k)=w_k$, $k=1,..., m$. Define $Q_1(z)=A_1\prod_{i=1}^{M_1}(z-a_{i1})\prod_{i=M_1+1}^{m-1}(1-\bar{a}_i z)$ and set $u_k=d_k^{2/p}w_k$. Then $Q_1(z_k)=u_k\prod_{i=1}^{m-1}(1-\bar{a}_{i1}z_k)^{1-(2/p)}$.

Let Q_2 be the unique polynomial of degree $m-1$ satisfying

$$Q_2(z_k)=u_k\prod_{i=1}^{m-1}(1-\bar{a}_{i1}z_k),$$

say,

$$Q_2(z)=A_2B_2^{(a)}(z)\prod_{i=1}^{m-1}(1-\bar{a}_{i2}z).$$

Then, for

$$\begin{aligned} g_2(z) &= Q_2(z)\left[\prod_{i=1}^{m-1}(1-\bar{a}_{i2}z)\right]^{(2/p)-1}\Big/\prod_{j=1}^{m}(1-\bar{z}_j z)^{2/p} \\ &= A_2B_2^{(a)}(z)\prod_{i=1}^{m-1}(1-\bar{a}_{i2}z)^{2/p}\Big/\prod_{j=1}^{m}(1-\bar{z}_j z)^{2/p}, \end{aligned}$$

we have

$$g_2(z_k)=w_k\prod_{i=1}^{m-1}(1-\bar{a}_{i2}z_k)^{(2/p)-1}\prod_{i=1}^{m-1}(1-\bar{a}_{i1}z_k).$$

Let Q_n be the polynomial of degree $m-1$ satisfying

$$Q_n(z_k)=u_k\prod_{i=1}^{m-1}(1-\bar{a}_{i,n-1}z_k)\Big/\prod_{i=1}^{m-1}(1-\bar{a}_{i,n-2}z_k),$$

say

$$Q_n(z) = A_n B_n^{(a)}(z) \prod_{i=1}^{m-1} (1 - \bar{a}_{in} z).$$

For

$$g_n(z) = Q_n(z) \prod_{i=1}^{m-1} (1 - \bar{a}_{in} z)^{(2/p)-1} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p}$$

$$= A_n B_n^{(a)}(z) \prod_{i=1}^{m-1} (1 - \bar{a}_{in} z)^{2/p} \Big/ \prod_{j=1}^{m} (1 - \bar{z}_j z)^{2/p},$$

we have

$$g_n(z_k) = w_k \prod_{i=1}^{m-1} (1 - \bar{a}_{in} z_k)^{(2/p)-1} \prod_{i=1}^{m-1} (1 - \bar{a}_{i,n-1} z_k)$$

THEOREM 4. *Suppose, for $n = n_\nu$,*

$$\left(\prod_{i=1}^{m-1} (1 - \bar{a}_{in} z_k) \Big/ \prod_{i=1}^{m-1} (1 - \bar{a}_{i,n-1} z_k) \right)_{n=1}^{\infty}$$

converges to 1, *for $k = 1,\ldots, m$. Let $a_i = \lim_{\nu\to\infty} a_{in_\nu} = \lim_{\nu\to\infty} a_{i,n_\nu-1}$. Then the corresponding $(Q_{n_\nu}(z))_{\nu=1}^{\infty}$ converges to $Q(z)$, the unique polynomial of degree $m - 1$ satisfying $Q(z_k) = w_k \prod (1 - \bar{z}_j z_k)^{2/p}$, $k = 1,\ldots, m$. Also, the subsequence $(g_{n_\nu}(z))$ converges to*

$$\tilde{g}(z) = A \prod_{i=1}^{M} [(z - a_i)/(1 - \bar{a}_i z)] \left[\prod_{i=1}^{m-1} (1 - \bar{a}_i z) \right]^{2/p} \Big/ \left[\prod_{j=1}^{m} (1 - z_j z) \right]^{2/p},$$

where $\tilde{g}(z_k) = w_k \prod_{i=1}^{m-1} (1 - \bar{a}_i z_k)^{(2/p)-1}$.

We note that, under the given hypothesis, the sequence of functions $(Q_{n_\nu}(z)/\prod(1 - \bar{z}_j z)^{2/p})_{\nu=1}^{\infty}$ converges to the best approximant in H_2 (in the L_2-sense) to $f^*(z)$.

5. POSSIBLE ALTERNATIVE PROCEDURES

In practice, the author recommends a preliminary trial step. If the zeros of f^* all lie exterior to $|z| = 1$, the function f^* is obtained immediately. Let P_1 be the polynomial of degree $m - 1$ satisfying $P_1(z_k) = w_k^{p/2} d_k$. If all the zeros of $P_1(z)$ are exterior to $|z| = 1$, we may write $P_1(z)$ in the form $C\prod_{i=1}^{m-1}(1 - \bar{c}_i z)$. Then, for

$$\tilde{f}(z) = C^{2/p}\left[\prod (1 - \bar{c}_i z)\right]^{2/p}\left[\prod_{j=1}^{m}(1 - \bar{z}_j z)\right]^{2/p},$$

we have $\tilde{f}(z_k) = w_k$, $k = 1,..., m$. By Theorem 1, $f^* = \tilde{f}$.

We note that, if $P_1(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1}$, with $a_0 > a_1 > \cdots > a_{m-1} > 0$, then all zeros of $P_1(z)$ lie exterior to $|z| = 1$ [6, p. 42, problem 2.1, Enestrom's Theorem]. If, however, $0 < a_0 < a_1 < \cdots < a_{m-1}$, then, by an application of Enestrom's Theorem [6, p. 224, problem 12], all zeros of P_n lie interior to $|z| = 1$. In case only a few of the zeros of f^* lie interior to $|z| = 1$, especially if p is near 2, it might be better to require, in the first step, that $P_1(z_k) = w_k^{p/2} d_k$. Then

$$[P_1(z)]^{2/p} = C^{2/p}\left[\prod_{i=1}^{M}(z - c_i)\right]^{2/p}\left[\prod_{i=M+1}^{m-1}(1 - \bar{c}_i z)\right]^{2/p}.$$

For

$$f_1(z) = C^{2/p}\prod_{i=1}^{M}[(z - c_i)/(1 - \bar{c}_i z)]\prod_{i=1}^{m-1}(1 - \bar{c}_i z)^{2/p}\Big/\prod_{j=1}^{m}(1 - \bar{z}_j z)^{2/p},$$

we have

$$\begin{aligned} f_1(z_k) &= [(P_1(z_k))^{2/p}/d_k^{2/p}][B^{(c)}(z_k)]^{1-2/p} \\ &= w_k\prod_{i=1}^{M}[(z_k - c_{i1})/(1 - \bar{c}_{i1} z_k)]^{1-(2/p)}. \end{aligned}$$

6. Some Related Extremal Functions

Given L^*, let $L(f^*) = \{g\colon g \in H_p \text{ and } g(z_j) = w_j/B^{(c)}(z_j),\ j = 1,..., m\}$ where $B^{(c)}(z)$ is the Blaschke product factor $\prod_i(z - c_i)/(1 - \bar{c}_i z)$ of f^*, extremal for L^*. The following result is immediate.

THEOREM 5. *The extremal function f^* for $L(f^*)$ is just $f^*/B^{(c)}$.*

In the following, $1 \leqslant p \leqslant \infty$ and $1 \leqslant p' \leqslant \infty$, with p and p' unrelated, except as specified. Suppose $L^* = L_p^*$ is given, and let $L_{p'}^* = \{g\colon g \in H_p,\ g(z_j) = w_j^{(p/p')}\}$. Then L_p^* and $L_{p'}^*$ are said to *correspond.*

Corollary 6.1 below indicates that, for any assigned z_j, $j = 1,..., m$, with $|z_j| < 1$, there exist corresponding functional values w_j such that, for L_p^* defined as $\{f\colon f \in H_p \text{ and } f(z_j) = w_j\}$, the extremal f_p^* is known. For, given any set of values u_j, the extremal element f_2^* for L_2^* defined as $\{f\colon f \in H_2,$ and $f(z_j) = u_j,\ j = 1,..., m\}$ is a rational function of the form $B_2(z)\, r(z)$, where all the zeros of the rational function $r(z)$ are outside $|z| = 1$, and where $B_2(z)$ is a Blaschke product. Then the extremal function $g_2{}^*$ for $L_2(f_2{}^*) = \{g: g \in H_2 \text{ and } g(z_j) = u_j/B_2(z_j),\ j = 1,..., m\}$ is just $r(z)$. Now, with $f(z_j) = w_j = (u_j/B_2(z_j))^{2/p}$, $j = 1,..., m$, the extremal element $f_p{}^*$ for $L_p{}^*$ is just $[r(z)]^{2/p}$.

THEOREM 6. *Suppose the extremal element g^* for $L_{p'}^*$ is nonvanishing. If $p = np'$, where n is a positive integer, then the extremal element $f_p{}^*$ for $L_p{}^*$ corresponding to $L_{p'}^*$ is just $(g^*)^{1/n}$.*

Proof. We have $f^{*p/p'} \in L_{p'}^*$. Since g^* is nonvanishing, evidently $g^{*(p'/p)} \in L_p{}^*$. Then

$$\|f_p{}^*\|_p^p \leqslant \|g_{p'}^{*(p'/p)}\|_p^p \leqslant \|f_p^{*(p/p')}\|_{p'}^{p'} = \|f_p{}^*\|_p^p,$$

yielding $\|g^{*p'/p}\|_{p'}^{p'} = \|f^*\|_p^p$.

COROLLARY 6.1. *If p is an even positive integer and if the extremal element g^* for $L_2{}^*$ is nonvanishing, then the extremal element $f_p{}^*$ for the corresponding $L_p{}^*$ is $g_2^{*2/p}$.*

The proof of the preceding theorem proves also

THEOREM 7. *Suppose $L_{p'}^*$ and $L_p{}^*$ correspond and that the respective extremal elements are $f_{p'}^*$ and $g_p{}^*$. If neither f^* nor g^* vanishes for $|z| < 1$, then $f^* = g^{*(p'/p)}$.*

Theorem 1 insures that examples may be constructed for which the minimizing functions satisfy the hypothesis of Theorem 6 or Theorem 7.

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